Homework 3 Solution.

1. Given an ordered basis β for a finite-dimensional vector space V over field F, show that the mapping T defined below is linear.

$$\begin{aligned} T : V &\longrightarrow F^n \\ \vec{x} &\longmapsto [\vec{x}]_\beta \end{aligned}$$

That is to prove $[a\vec{x} + \vec{y}]_\beta = a[\vec{x}]_\beta + [\vec{y}]_\beta$ for any $\vec{x},\ \vec{y} \in V$ and $a \in F$.

Proof: Let $\beta = \{v_1, \cdots v_n\}$ be an ordered basis for V. over F

$\forall \vec{x}, \vec{y} \in V.\quad \forall a \in F$

$\exists!\ \alpha_1 \cdots \alpha_n,\ \gamma_1 \cdots \gamma_n \in F$ such that

$$\vec{x} = \sum_1^n \alpha_i \cdot v_i \qquad \vec{y} = \sum_1^n \gamma_i \cdot v_i$$

Then $$[\vec{x}]_\beta = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in F^n \qquad [\vec{y}]_\beta = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \in F^n$$

Note that
$$\begin{aligned} a\vec{x} + \vec{y} &= a \cdot \left(\sum_1^n \alpha_i \cdot v_i\right) + \left(\sum_1^n \gamma_i \cdot v_i\right) \\ &= \sum_1^n (a \cdot \alpha_i + \gamma_i) \cdot v_i \end{aligned}$$

Thus
$$[a \cdot \vec{x} + \vec{y}]_\beta = \begin{pmatrix} a \cdot \alpha_1 + \gamma_1 \\ \vdots \\ a\alpha_n + \gamma_n \end{pmatrix} = a \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} + \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} = a[\vec{x}]_\beta + [\vec{y}]_\beta$$

i.e. $T(a\vec{x} + \vec{y}) = a \cdot T(\vec{x}) + T(\vec{y})$

2. Sec. 2.1: Q31

Definitions. *Let* V *be a vector space, and let* $\mathsf{T}: \mathsf{V} \to \mathsf{V}$ *be linear. A subspace* W *of* V *is said to be* **T-invariant** *if* $\mathsf{T}(x) \in \mathsf{W}$ *for every* $x \in \mathsf{W}$, *that is,* $\mathsf{T}(\mathsf{W}) \subseteq \mathsf{W}$. *If* W *is* T*-invariant, we define the* **restriction of** T **on** W *to be the function* $\mathsf{T_W}: \mathsf{W} \to \mathsf{W}$ *defined by* $\mathsf{T_W}(x) = \mathsf{T}(x)$ *for all* $x \in \mathsf{W}$.

31. Suppose that $\mathsf{V} = \mathsf{R}(\mathsf{T}) \oplus \mathsf{W}$ and W is T-invariant. (Recall the definition of *direct sum* given in the exercises of Section 1.3.)

(a) Prove that $\mathsf{W} \subseteq \mathsf{N}(\mathsf{T})$.

(b) Show that if V is finite-dimensional, then $\mathsf{W} = \mathsf{N}(\mathsf{T})$.

(c) Show by example that the conclusion of (b) is not necessarily true if V is not finite-dimensional.

Solution.

(a) $\forall x \in W \subset V$. $T(x) \in R(T)$

Besides, W is T-invariant, so $T(x) \in W$.

Thus $T(x) \in R(T) \cap W$

Since $V = R(T) \oplus W$, we have $R(T) \cap W = \{0\}$

Thus $T(x) = 0$ i.e. $x \in N(T)$

Therefore $W \subset N(T)$

(b) Since V is finite-dim. by rank-nullity thm,

$\dim(V) = \dim(R(T)) + \dim(N(T))$

Since $V = R(T) \oplus W$, we have

$\dim(V) = \dim(R(T)) + \dim(W)$

Thus. $\dim(N(T)) = \dim(W)$

Since $W \subset N(T)$, we have $N(T) = W$

(c) Let $V = C^\infty(\mathbb{R})$ $\begin{cases} \forall f, g \in V. \quad a \in \mathbb{R} \\ (af+g)(x) := a \cdot f(x) + g(x) \end{cases}$

$W = \{0\}$

$T: V \to V$ is defined as $T(f) = f'$

① W is T-invariant ② $R(T) = V$ therefore $V = R(T) \oplus W$

But $N(T)$ is the collection of constant functions. $N(T) \neq W$

3. Sec. 2.1: Q32

32. Suppose that W is T-invariant. Prove that $N(T_W) = N(T) \cap W$ and $R(T_W) = T(W)$.

Proof: W is T-invariant

$T_W: W \to W \quad T_W(x) = T(x) \quad \forall x \in W.$

- $\forall x \in N(T_W) \subset W$

  $T(x) = T_W(x) = 0$ then $x \in N(T)$

  Since $x \in W$, $x \in N(T) \cap W$ i.e. $N(T_W) \subset N(T) \cap W$

- $\forall x \in N(T) \cap W$

  $\begin{cases} x \in N(T) \Rightarrow T(x) = 0 \\ x \in W \Rightarrow T_W(x) = T(x) \end{cases} \Rightarrow T_W(x) = 0 \Rightarrow x \in N(T_W)$

  Thus $N(T) \cap W \subset N(T_W)$

- $\forall y \in R(T_W)$

  $\exists x \in W$ st $y = T_W(x) = T(x) \in T(W)$

  Thus $R(T_W) \subset T(W)$

- $\forall y \in T(W)$.

  $\exists x \in W$ st $y = T(x) = T_W(x) \in R(T_W)$

  Thus $T(W) \subset R(T_W)$

4. Sec. 2.2: Q4

4. Define

$$\mathsf{T}\colon \mathsf{M}_{2\times 2}(R) \to \mathsf{P}_2(R) \quad \text{by} \quad \mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2.$$

Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad \gamma = \{1, x, x^2\}.$$

Compute $[\mathsf{T}]_\beta^\gamma$.

Solution.

$$T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = 1 = 1 + 0\cdot x + 0\cdot x^2$$

$$T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = 1 + x^2 = 1 + 0\cdot x + 1\cdot x^2$$

$$T\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = 0 = 0 + 0\cdot x + 0\cdot x^2$$

$$T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = 2x = 0 + 2\cdot x + 0\cdot x^2$$

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

5. Sec. 2.2: Q12

12. Let V be a finite-dimensional vector space and T be the projection on W along W′, where W and W′ are subspaces of V. (See the definition in the exercises of Section 2.1 on page 76.) Find an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

Proof: $V = W \oplus W'$

$\forall v \in V.\ \exists!\ w \in W\ \ !\ w' \in W'$ s.t $v = w + w'$

$T: V \to V$

$v \mapsto w.$

V is finite-dim. so W and W′ are finite-dim.

Let $\gamma = \{u_1 \cdots u_n\}$ be a basis for W

$\gamma' = \{v_1 \cdots v_m\}$ be a basis for W′

☆ Note that $\beta = \gamma \cup \gamma'$ is a basis for $W \oplus W' = V$

$\beta = \{u_1 \cdots u_n\ v_1 \cdots v_m\}$ is the ordered basis for V.

$$\begin{cases} T(u_i) = u_i & i = 1 \cdots n \\ T(v_j) = 0 & j = 1 \cdots m \end{cases}$$

Thus $[T]_\beta = \begin{bmatrix} I_n & O_{n\times m} \\ O_{m\times n} & O_{m\times m} \end{bmatrix}$ is diagonal